情報理論レポート 2(各種情報量)解答例

1．各種情報量

(1) 各候補者が当選する確率を求め、当選に関する事象系 R を求めよ。

事象 $\alpha \in A$ と事象 $\beta \in B$ の積事象の確率 $P(A=\alpha, B=\beta)$ と当選者を次の表に示す。

$P(\alpha,\beta)$	$A(2,0,0)$	$A(0,2,0)$	$A(0,0,2)$	$A(1,1,0)$	$A(1,0,1)$	$A(0,1,1)$
$B(1,0,0)$	$r_1, \frac{4}{12}\bullet\frac{1}{8}$	$r_2, \frac{2}{12}\bullet\frac{1}{8}$	$r_3, \frac{1}{12}\bullet\frac{1}{8}$	$r_1, \frac{2}{12}\bullet\frac{1}{8}$	$r_1, \frac{2}{12}\bullet\frac{1}{8}$	$r_1, \frac{1}{12}\bullet\frac{1}{8}$
$B(0,1,0)$	$r_1, \frac{4}{12}\bullet\frac{3}{8}$	$r_2, \frac{2}{12}\bullet\frac{3}{8}$	$r_3, \frac{1}{12}\bullet\frac{3}{8}$	$r_2, \frac{2}{12}\bullet\frac{3}{8}$	$r_1, \frac{2}{12}\bullet\frac{3}{8}$	$r_2, \frac{1}{12}\bullet\frac{3}{8}$
$B(0,0,1)$	$r_1, \frac{4}{12}\bullet\frac{4}{8}$	$r_2, \frac{2}{12}\bullet\frac{4}{8}$	$r_3, \frac{1}{12}\bullet\frac{4}{8}$	$r_1, \frac{2}{12}\bullet\frac{4}{8}$	$r_3, \frac{2}{12}\bullet\frac{4}{8}$	$r_3, \frac{1}{12}\bullet\frac{4}{8}$

この表より、以下のように確率の計算ができる。

$$P\left(R=r_1\right)=\left(\frac{4}{12}\bullet\frac{1}{8}+\frac{4}{12}\bullet\frac{3}{8}+\frac{4}{12}\bullet\frac{4}{8}\right)+\left(\frac{2}{12}\bullet\frac{1}{8}+\frac{2}{12}\bullet\frac{4}{8}\right)+\left(\frac{2}{12}\bullet\frac{1}{8}+\frac{2}{12}\bullet\frac{3}{8}\right)+\left(\frac{1}{12}\bullet\frac{1}{8}\right)$$
$$=\left(\frac{1}{3}\right)+\left(\frac{5}{48}\right)+\left(\frac{4}{48}\right)+\left(\frac{1}{96}\right)$$
$$=\frac{32+10+8+1}{96}$$
$$=\frac{51}{96}$$

$$P\left(R=r_2\right)=\left(\frac{2}{12}\bullet\frac{1}{8}+\frac{2}{12}\bullet\frac{3}{8}+\frac{2}{12}\bullet\frac{4}{8}\right)+\left(\frac{2}{12}\bullet\frac{3}{8}\right)+\left(\frac{1}{12}\bullet\frac{3}{8}\right)$$
$$=\left(\frac{1}{6}\right)+\left(\frac{3}{48}\right)+\left(\frac{3}{96}\right)$$
$$=\frac{16+6+3}{96}$$
$$=\frac{25}{96}$$

$$P\left(R=r_3\right)=\left(\frac{1}{12}\bullet\frac{1}{8}+\frac{1}{12}\bullet\frac{3}{8}+\frac{1}{12}\bullet\frac{4}{8}\right)+\left(\frac{2}{12}\bullet\frac{4}{8}\right)+\left(\frac{1}{12}\bullet\frac{4}{8}\right)$$
$$=\left(\frac{1}{12}\right)+\left(\frac{1}{12}\right)+\left(\frac{1}{24}\right)$$
$$=\frac{2+2+1}{24}$$
$$=\frac{20}{96}$$

以上より、

$$R=\begin{Bmatrix} r_1 & , & r_2 & , & r_3 \\ \dfrac{51}{96} & , & \dfrac{25}{96} & , & \dfrac{20}{96} \end{Bmatrix}$$

(2) $H(R), H(A), H(B)$を求めよ。

$$\begin{aligned}
H(R) &= -\sum_{\gamma\in R} P(\gamma)\log P(\gamma) \\
&= -\frac{51}{96}\log\frac{51}{96}-\frac{25}{96}\log\frac{25}{96}-\frac{20}{96}\log\frac{20}{96} \\
&= \frac{51}{96}\log\frac{96}{51}+\frac{25}{96}\log\frac{96}{25}+\frac{20}{96}\log\frac{96}{20} \\
&= \left(\frac{51}{96}+\frac{25}{96}+\frac{20}{96}\right)\log 96-\frac{51}{96}\log 51-\frac{25}{96}\log 25-\frac{20}{96}\log 20 \\
&\simeq 6.585-3.013-1.209-0.9004 \\
&\simeq 1.462 \quad [bit\ /\ \text{事象}]
\end{aligned}$$

$$\begin{aligned}
H(A) &= -\sum_{\alpha\in A} P(\alpha)\log P(\alpha) \\
&= -2\times\frac{1}{12}\log\frac{1}{12}-3\times\frac{2}{12}\log\frac{2}{12}-\frac{4}{12}\log\frac{4}{12} \\
&= \frac{2}{12}\log 12+\frac{6}{12}\left(\log 12-\log 2\right)+\frac{4}{12}\left(\log 12-\log 4\right) \\
&= \log 12-\frac{1}{2}-\frac{2}{3} \\
&= 2+\log 3-\frac{7}{6} \\
&\simeq 2.418 \quad [bit\ /\ \text{事象}]
\end{aligned}$$

$$\begin{aligned}
H(B) &= -\sum_{\beta\in B} P(\beta)\log P(\beta) \\
&= -\frac{1}{8}\log\frac{1}{8}-\frac{3}{8}\log\frac{3}{8}-\frac{4}{8}\log\frac{4}{8} \\
&= \frac{1}{8}\log 8+\frac{3}{8}\left(\log 8-\log 3\right)+\frac{4}{8}\left(\log 8-\log 4\right) \\
&= \log 8-\frac{3}{8}\log 3-1 \\
&= 2-\frac{3}{8}\log 3 \\
&\simeq 1.406 \quad [bit\ /\ \text{事象}]
\end{aligned}$$

(3) 次の条件付きエントロピーを求めよ。
$H(R \mid A), H(B \mid R), H(A \mid B)$

① $H(R \mid A)$

まず、次のように条件付き確率が求められる。

$P(R=\gamma \mid A=\alpha)$	$A(2,0,0)$	$A(0,2,0)$	$A(0,0,2)$	$A(1,1,0)$	$A(1,0,1)$	$A(0,1,1)$
r_1	1	0	0	$\dfrac{\frac{2}{96}+\frac{8}{96}}{\frac{2}{96}+\frac{6}{96}+\frac{8}{96}}$	$\dfrac{\frac{2}{96}+\frac{6}{96}}{\frac{2}{96}+\frac{6}{96}+\frac{8}{96}}$	$\dfrac{\frac{1}{96}}{\frac{1}{96}+\frac{3}{96}+\frac{4}{96}}$
r_2	0	1	0	$\dfrac{\frac{6}{96}}{\frac{2}{96}+\frac{6}{96}+\frac{8}{96}}$	0	$\dfrac{\frac{3}{96}}{\frac{1}{96}+\frac{3}{96}+\frac{4}{96}}$
r_3	0	0	1	0	$\dfrac{\frac{8}{96}}{\frac{2}{96}+\frac{6}{96}+\frac{8}{96}}$	$\dfrac{\frac{4}{96}}{\frac{1}{96}+\frac{3}{96}+\frac{4}{96}}$

よって、各要素$\alpha \in A$に対する条件付きエントロピー$H(R \mid \alpha)$が次のように求められる。
$H(R \mid A(2,0,0)) = 0$
$H(R \mid A(0,2,0)) = 0$
$H(R \mid A(0,0,2)) = 0$

$$H(R \mid A(1,1,0)) = -\frac{5}{8}\log\frac{5}{8} - \frac{3}{8}\log\frac{3}{8} = \mathcal{H}\left(\frac{5}{8}\right) \simeq 0.9544$$

$$H(R \mid A(1,0,1)) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \mathcal{H}\left(\frac{1}{2}\right) = 1.0$$

$$H(R \mid A(0,1,1)) = -\frac{1}{8}\log\frac{1}{8} - \frac{3}{8}\log\frac{3}{8} - \frac{4}{8}\log\frac{4}{8}$$
$$= \log 8 - \frac{3}{8}\log 3 - \frac{4}{8}\log 4$$
$$\simeq 2 - 0.5944$$
$$\simeq 1.406$$

以上より、次のように$H(R \mid A)$が計算できる。

$$
\begin{aligned}
&H(R \mid A)=\sum_{\alpha\in A}P\left(\alpha\right)H\left(R \;\middle|\; \alpha\right)\\
&\simeq \frac{4}{12}\bullet 0+\frac{2}{12}\bullet 0+\frac{1}{12}\bullet 0+\frac{2}{12}\bullet\left(0.9544\right)+\frac{2}{12}\bullet\left(1.0\right)+\frac{1}{12}\bullet\left(1.406\right)\\
&\simeq 0.4429 \quad [bit\ /\ \text{事象}]
\end{aligned}
$$

② $H(B \mid R)$

まず、次のように条件付き確率が求められる。

$$
P\left(B\left(1,0,0\right) \mid r_1\right)=\frac{\displaystyle\sum_{\alpha\in A,\beta=B(1,0,0),\gamma=r_1} P(\alpha,B(1,0,0))}{\displaystyle\sum_{\alpha\in A,\beta\in B,\gamma=r_1} P(\alpha,\beta)}=\frac{\frac{4}{96}+\frac{2}{96}+\frac{2}{96}+\frac{1}{96}}{\frac{51}{96}}=\frac{9}{51}
$$

$$
P\left(B\left(0,1,0\right) \mid r_1\right)=\frac{\frac{12}{96}+\frac{6}{96}}{\frac{51}{96}}=\frac{18}{51}
$$

$$
P\left(B\left(0,0,1\right) \mid r_1\right)=\frac{\frac{16}{96}+\frac{8}{96}}{\frac{51}{96}}=\frac{24}{51}
$$

$$
P\left(B\left(1,0,0\right) \mid r_2\right)=\frac{\displaystyle\sum_{\alpha\in A,\beta=B(1,0,0),\gamma=r_2} P(\alpha,B(1,0,0))}{P\left(r_2\right)}=\frac{\frac{2}{96}}{\frac{25}{96}}=\frac{2}{25}
$$

$$
P\left(B\left(0,1,0\right) \mid r_2\right)=\frac{\frac{6}{96}+\frac{6}{96}+\frac{3}{96}}{\frac{25}{96}}=\frac{15}{25}
$$

$$
P\left(B\left(0,0,1\right) \mid r_2\right)=\frac{\frac{8}{96}}{\frac{25}{96}}=\frac{8}{25}
$$

$$
P\left(B\left(1,0,0\right) \mid r_3\right)=\frac{\frac{1}{96}}{\frac{20}{96}}=\frac{1}{20}
$$

$$P\left(B\left(0,1,0\right)\mid r_3\right)=\frac{\frac{3}{96}}{\frac{20}{96}}=\frac{3}{20}$$

$$P\left(B\left(0,0,1\right)\mid r_3\right)=\frac{\frac{4}{96}+\frac{8}{96}+\frac{4}{96}}{\frac{20}{96}}=\frac{16}{20}$$

よって、各要素$\gamma\in R$に対する条件付きエントロピー$H(B\mid\gamma)$が次のように求められる。

$$\begin{aligned}
H\left(B\mid r_1\right)&=-\frac{9}{51}\log\frac{9}{51}-\frac{18}{51}\log\frac{18}{51}-\frac{24}{51}\log\frac{24}{51}\\
&=\log 51-\frac{9}{51}\log 9-\frac{18}{51}\log 18-\frac{24}{51}\log 24\\
&=\log 51-\frac{2\times 9+2\times 18+24}{51}\log 3-\frac{18+3\times 24}{51}\\
&\simeq 5.672-2.424-1.765\\
&\simeq 1.483\quad [bit/\text{事象}]
\end{aligned}$$

$$\begin{aligned}
H\left(B\mid r_2\right)&=-\frac{2}{25}\log\frac{2}{25}-\frac{15}{25}\log\frac{15}{25}-\frac{8}{25}\log\frac{8}{25}\\
&=\log 25-\frac{2}{25}-\frac{15}{25}\left(\log 3+\log 5\right)-\frac{24}{25}\\
&=\frac{35}{25}\log 5-\frac{15}{25}\log 3-\frac{26}{25}\\
&\simeq 3.251-0.951-1.04\\
&\simeq 1.260\quad [bit/\text{事象}]
\end{aligned}$$

$$\begin{aligned}
H\left(B\mid r_3\right)&=-\frac{1}{20}\log\frac{1}{20}-\frac{3}{20}\log\frac{3}{20}-\frac{16}{20}\log\frac{16}{20}\\
&=\log 20-\frac{3}{20}\log 3-\frac{16}{20}\times 4\\
&=\left(2-\frac{16}{5}\right)+\log 5-\frac{3}{20}\log 3\\
&\simeq -1.2+2.322-0.2377\\
&\simeq 0.884\quad [bit/\text{事象}]
\end{aligned}$$

以上より、次のように$H(B\mid R)$が計算できる。

$$
\begin{aligned}
&H(B \mid R) = \sum_{\gamma \in R} P\left(\gamma\right) H\left(B \mid \gamma\right) \\
&\simeq \frac{51}{96} \bullet \left(1.483\right) + \frac{25}{96} \bullet \left(1.260\right) + \frac{20}{96} \bullet \left(0.884\right) \\
&\simeq 1.30 \quad [bit \ / \ \text{事象}]
\end{aligned}
$$

③ $H(A \mid B)$

次のように条件付き確率が計算できる。

$P(A=\alpha \mid B=\beta)$	$A(2,0,0)$	$A(0,2,0)$	$A(0,0,2)$	$A(1,1,0)$	$A(1,0,1)$	$A(0,1,1)$
$B(1,0,0)$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$
$B(0,1,0)$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$
$B(0,0,1)$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

よって、

$$
\begin{aligned}
&H\left(A \mid B(1,0,0)\right) = -\sum_{\alpha \in A} P\left(\alpha \mid B(1,0,0)\right) \log P\left(\alpha \mid B(1,0,0)\right) \\
&= -2 \times \frac{1}{12} \log \frac{1}{12} - 3 \times \frac{2}{12} \log \frac{2}{12} - \frac{4}{12} \log \frac{4}{12} \\
&= H(A)
\end{aligned}
$$

$$
H\left(A \mid B(0,1,0)\right) = H(A)
$$

$$
H\left(A \mid B(0,1,0)\right) = H(A)
$$

以上より、次のように計算できる。

$$
\begin{aligned}
&H(A \mid B) = \frac{1}{8} H(A \mid B\left(1,0,0\right)) + \frac{3}{8} H(A \mid B\left(0,1,0\right)) + \frac{4}{8} H(A \mid B\left(0,0,1\right)) \\
&= \frac{1}{8} H(A) + \frac{3}{8} H(A) + \frac{4}{8} H(A) \\
&= H(A) \\
&\simeq 2.418 \quad [bit \ / \ \text{事象}]
\end{aligned}
$$

☆なお、事象系 A と事象系 B は独立なので、

$H(A|B) = H(A)$

$H(B|A) = H(B)$

が成り立つ。

(4)次の結合エントロピーを求めよ。
$H(R,A), H(B,R), H(A,B)$

① $H(R,A)$

事象 $\alpha \in A$ と事象 $\gamma \in R$ の同時確率 $P(\alpha,\gamma)$ は次のように求められる。

$P(A=\alpha, R=\gamma)$	$A(2,0,0)$	$A(0,2,0)$	$A(0,0,2)$	$A(1,1,0)$	$A(1,0,1)$	$A(0,1,1)$
r_1	$\frac{4+12+16}{96}$	0	0	$\frac{2+8}{96}$	$\frac{2+6}{96}$	$\frac{1}{96}$
r_2	0	$\frac{2+6+8}{96}$	0	$\frac{6}{96}$	0	$\frac{3}{96}$
r_3	0	0	$\frac{1+3+4}{96}$	0	$\frac{8}{96}$	$\frac{4}{96}$

よって、$H(R,A)$ は次のように計算できる。

$$H(R,A) = -\sum_{\alpha\in A,\gamma\in R} P(\alpha,\gamma)\log P(\alpha,\gamma)$$

$$= -\frac{32}{96}\log\frac{32}{96} - \frac{16}{96}\log\frac{16}{96} - \frac{8}{96}\log\frac{8}{96} - \frac{10}{96}\log\frac{10}{96} - \frac{6}{96}\log\frac{6}{96}$$
$$-\frac{8}{96}\log\frac{8}{96} - \frac{8}{96}\log\frac{8}{96} - \frac{1}{96}\log\frac{1}{96} - \frac{3}{96}\log\frac{3}{96} - \frac{4}{96}\log\frac{4}{96}$$
$\simeq 2.86 \quad [bit$ / 事象$]$

② $H(B,R)$

事象 $\beta \in B$ と事象 $\gamma \in R$ の同時確率 $P(\alpha,\gamma)$ は次のように求められる。

$P(B=\beta, R=\gamma)$	r_1	r_2	r_3
$B(1,0,0)$	$\frac{4+2+2+1}{96}$	$\frac{2}{96}$	$\frac{1}{96}$
$B(0,1,0)$	$\frac{12+6}{96}$	$\frac{6+6+3}{96}$	$\frac{3}{96}$
$B(0,0,1)$	$\frac{16+8}{96}$	$\frac{8}{96}$	$\frac{4+8+4}{96}$

よって、$H(B,R)$ は次のように計算できる。

$$H(B,R) = -\sum_{\beta \in B, \gamma \in R} P(\beta,\gamma)\log P(\beta,\gamma)$$
$$= -\frac{9}{96}\log\frac{9}{96} - \frac{2}{96}\log\frac{2}{96} - \frac{1}{96}\log\frac{1}{96}$$
$$-\frac{18}{96}\log\frac{18}{96} - \frac{15}{96}\log\frac{15}{96} - \frac{3}{96}\log\frac{3}{96}$$
$$-\frac{24}{96}\log\frac{24}{96} - \frac{8}{96}\log\frac{8}{96} - \frac{16}{96}\log\frac{16}{96}$$
$$\simeq 2.762 \quad [bit\,/\,\text{事象}]$$

③ $H(A,B)$

$$H(A,B) = -\sum_{\alpha \in A, \beta \in B} P(\alpha,\beta)\log P(\alpha,\beta)$$
$$= -\frac{4}{12}\bullet\frac{1}{8}\log\frac{4}{12}\bullet\frac{1}{8} - \frac{4}{12}\bullet\frac{3}{8}\log\frac{4}{12}\bullet\frac{3}{8} - \frac{4}{12}\bullet\frac{4}{8}\log\frac{4}{12}\bullet\frac{4}{8}$$
$$-\frac{2}{12}\bullet\frac{1}{8}\log\frac{2}{12}\bullet\frac{1}{8} - \frac{2}{12}\bullet\frac{3}{8}\log\frac{2}{12}\bullet\frac{3}{8} - \frac{2}{12}\bullet\frac{4}{8}\log\frac{2}{12}\bullet\frac{4}{8}$$
$$-\frac{1}{12}\bullet\frac{1}{8}\log\frac{1}{12}\bullet\frac{1}{8} - \frac{1}{12}\bullet\frac{3}{8}\log\frac{1}{12}\bullet\frac{3}{8} - \frac{1}{12}\bullet\frac{4}{8}\log\frac{1}{12}\bullet\frac{4}{8}$$
$$-\frac{2}{12}\bullet\frac{1}{8}\log\frac{2}{12}\bullet\frac{1}{8} - \frac{2}{12}\bullet\frac{3}{8}\log\frac{2}{12}\bullet\frac{3}{8} - \frac{2}{12}\bullet\frac{4}{8}\log\frac{2}{12}\bullet\frac{4}{8}$$
$$-\frac{2}{12}\bullet\frac{1}{8}\log\frac{2}{12}\bullet\frac{1}{8} - \frac{2}{12}\bullet\frac{3}{8}\log\frac{2}{12}\bullet\frac{3}{8} - \frac{2}{12}\bullet\frac{4}{8}\log\frac{2}{12}\bullet\frac{4}{8}$$
$$-\frac{1}{12}\bullet\frac{1}{8}\log\frac{1}{12}\bullet\frac{1}{8} - \frac{1}{12}\bullet\frac{3}{8}\log\frac{1}{12}\bullet\frac{3}{8} - \frac{1}{12}\bullet\frac{4}{8}\log\frac{1}{12}\bullet\frac{4}{8}$$
$$= \frac{4}{12}H(B) - \frac{4}{12}\log\frac{4}{12}$$
$$+\frac{2}{12}H(B) - \frac{2}{12}\log\frac{2}{12}$$
$$+\frac{1}{12}H(B) - \frac{1}{12}\log\frac{1}{12}$$
$$+\frac{2}{12}H(B) - \frac{2}{12}\log\frac{2}{12}$$
$$+\frac{2}{12}H(B) - \frac{2}{12}\log\frac{2}{12}$$
$$+\frac{2}{12}H(B) - \frac{1}{12}\log\frac{1}{12}$$
$$= H(B) + H(A)$$
$$\simeq 1.406 + 2.418$$
$$\simeq 3.824 \quad [bit\,/\,\text{事象}]$$

☆なお、事象系 A と事象系 B は独立なので、

$H(A,B)=H(A)+H(B)=H(B)+H(A)=H(B,A)$

が成り立つ。

(4) 次の相互情報量を求めよ。

$I(R;A), I(B,R), I(A;B)$

① $I(R;A)$

$I(R;A)=H(R)-H(R\mid A)$

$\simeq 1.462-0.442$

$\simeq 1.02 \quad [bit/\text{事象}]$

なお、次の計算でも同じ値になるはず。

$I(R;A)=H(R)+H(A)-H(R,A)$

$\simeq 1.462+2.418-2.86$

$\simeq 1.02 \quad [bit/\text{事象}]$

② $I(B,R)$

$I(B,R)=H(B)-H(B\mid R)$

$\simeq 1.406-1.30$

$\simeq 0.106 \quad [bit/\text{事象}]$

なお、次の計算でも同じ値になるはず。

$I(B;R)=H(B)+H(R)-H(B,R)$

$\simeq 1.406+1.462-2.762$

$\simeq 0.106 \quad [bit/\text{事象}]$

③ $I(A;B)$

$I(A;B)=H(A)-H(A\mid B)$

$=H(B)-H(B\mid A)$

$=I(B;A)$

$=0$

☆独立な事象系どうしでは、相互情報量は０になる。